

# WELL-POSEDNESS AND INVERSE ROBIN ESTIMATE FOR A MULTISCALE ELLIPTIC/PARABOLIC SYSTEM

MARTIN LIND, ADRIAN MUNTEAN, AND OMAR RICHARDSON

**ABSTRACT.** We establish the well-posedness of a coupled micro-macro parabolic-elliptic system modeling the interplay between two pressures in a gas-liquid mixture close to equilibrium that is filling a porous media with distributed microstructures. Additionally, we prove a local stability estimate for the inverse micro-macro Robin problem, potentially useful in identifying quantitatively a micro-macro interfacial Robin transfer coefficient given microscopic measurements on accessible fixed interfaces. To tackle the solvability issue we use two-scale energy estimates and two-scale regularity/compactness arguments cast in the Schauder's fixed point theorem. A number of auxiliary problems, regularity, and scaling arguments are used in ensuring the suitable Fréchet differentiability of the solution and the structure of the inverse stability estimate.

## 1. INTRODUCTION

We are interested in developing evolution equations able to describe multiscale spatial interactions in gas-liquid mixtures, targeting a rigorous mathematical justification of Richards-like equations - upscaled model equations generally chosen in a rather *ad hoc* manner by the engineering communities to describe the motion of flow in unsaturated porous media. The main issue is that one lacks a rigorous derivation of the Darcy's law for such flow (see Hornung [2012] (chapter 1) for a derivation via periodic homogenization techniques of the Darcy law for the saturated case).

If air-water interfaces can be assume to be stagnant for a reasonable time span, then averaging techniques for materials with locally periodic microstructures (compare e.g. Chechkin and Piatnitski [1998]) lead in suitable scaling regimes to what we refer here as *two-pressure evolution systems*. These are normally coupled parabolic-elliptic systems responsible for the joint evolution in time  $t \in (0, T)$  ( $T < +\infty$ ) of a parameter-dependent *microscopic pressure*  $R\rho(t, x, y)$  evolving with respect to  $y \in Y \subset \mathbb{R}^d$  for any given macroscopic spatial position  $x \in \Omega$  and a *macroscopic pressure*  $\pi(t, x)$  with  $x \in \Omega$  for any given  $t$ . Here  $R$  denotes the universal constant of gases. The two-scale geometry we have in mind is depicted in Figure 1 below.

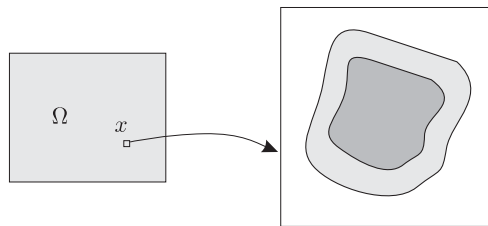


FIGURE 1. The macroscopic domain  $\Omega$  and microscopic pore  $Y$  at  $x \in \Omega$

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To cast the physical problem in mathematical terms as stated in (1), we need a number of dimensional constant parameters ( $A$  (gas permeability),  $D$  (diffusion coefficient for the gaseous species),  $p_F$  (atmospheric pressure),  $\rho_F$  (gas density)) and dimensional functions ( $k$  (Robin coefficient) and  $\rho_I$  (initial liquid density)). It is worth noting that excepting the Robin coefficient  $k$ , all the model parameters and functions are either known or can be accessed directly via measurements. Getting grip on a priori values of  $k$  is more intricate simply because this coefficient is defined on the Robin part of the boundary of  $\partial Y$ , say  $\Gamma_R$ , where the micro-macro information transfer takes actively place. The Neumann part of the boundary  $\Gamma_N := \partial Y - \Gamma_R$  is assumed to be accessible via measurements, while  $\Gamma_R$  is thought here as inaccessible.

Our aim is twofold:

- (1) ensure the well-posedness in a suitable sense of our two-pressure system with  $k$  taken to be known;
- (2) prove stability estimates with respect to  $k$  for the inverse micro-macro Robin problem ( $k$  is now unknown, but measured values of the microscopic pressure are available on  $\Gamma_N$ ).

The main results reported here are Theorem 3.1 (the weak solvability of (1)) and Theorem 5.1 (the local stability for the inverse micro-macro Robin problem).

The choice of problem and approach is in line with other investigations running for two-scale systems, or systems with distributed microstructures, like Lind and Muntean [2016], Meier [2008], Peszynska and Showalter [2007]. As far as we are aware, this is for the first time that an inverse Robin problem is treated in a two-scale setting. A remotely connected single-scale inverse Robin problem is treated in Nakamura and Wang [2015].

## 2. PROBLEM FORMULATION

We shall consider the following parabolic-elliptic problem posed on two spatial scales  $x \in \Omega$  and  $y \in Y$ .

$$\begin{cases} -A\rho_F\Delta_x\pi = f(\pi, \rho) & \text{in } \Omega \\ \partial_t\rho - D\Delta_y\rho = 0 & \text{in } \Omega \times Y \\ -D\nabla_y\rho \cdot n_y = k(\pi + p_F - R\rho) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y\rho \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ \pi = 0 & \text{at } \partial\Omega \\ \rho(t=0) = \rho_I(x, y) & \text{in } \Omega \times Y, \end{cases} \quad (1)$$

where the parameters, coefficients and the nonlinear function  $f$  satisfies the assumptions discussed below (see Section 2.1). The initial condition for  $p$  follows from the coupling between  $\rho$  and  $p$ .

A prominent role in this paper is played by the micro-macro Robin transfer coefficient  $k$ , which is selected from the following set

$$\mathcal{K} := \{k \in \mathcal{L}^2(\Gamma_R) : 0 < \underline{k} \leq k(y) \leq \bar{k} \text{ for } y \in \Gamma_R\}.$$

### 2.1. Assumptions.

- (A<sub>1</sub>) The domains  $\Omega, Y$  have Lipschitz continuous boundaries.
- (A<sub>2</sub>) The parameters satisfy  $A, D, \rho_F, R \in (0, \infty)$ .
- (A<sub>3</sub>) The initial value  $\rho_I \in H^1(\Omega \times Y)$ .
- (A<sub>4</sub>)  $f(\lambda u, \mu v) = \lambda^\alpha \mu^\beta f(u, v)$  where  $\alpha + \beta = 1$ ,  $\alpha, \beta > 0$ .

(A<sub>5</sub>) There is a structural constant  $C^* > 0$  such that

$$\int_{\Omega} |f(u_1, v) - f(u_2, v)|^2 dx \leq C^* \|u_1 - u_2\|_{\mathcal{L}^2(\Omega)}^2$$

uniformly in  $v \in \mathcal{L}^2(\Omega; \mathcal{L}^2(Y))$ .

(A<sub>6</sub>) There is a constant  $C > 0$  such that

$$\int_{\Omega} f(u, v)^2 dx \leq C \|v\|_{\mathcal{L}^2(\Omega; H^1(Y))}^2.$$

(A<sub>7</sub>) The constant  $C^*$  in (A<sub>5</sub>) satisfies

$$C^* c_P(\Omega) < 1,$$

where  $c_P(\Omega)$  is the Poincaré constant of the domain  $\Omega$  (see Proposition 2.1 below).

*Remark 1.* Assumptions (A<sub>1</sub>)-(A<sub>3</sub>) have clear geometrical or physical meanings, while (A<sub>4</sub>)-(A<sub>7</sub>) are technical. The assumption (A<sub>7</sub>) is only used when deriving uniqueness of the weak solution to (1). Note also that for some special classes of domains, the Poincaré constant can be quantitatively estimated, see e.g. Mikhlin [1981].

**2.2. Auxiliary results.** In this section, we state some auxiliary results that will be useful in this context.

**Proposition 2.1** (Poincaré's inequality). *Let  $\Omega \subset \mathbb{R}^d$  be a fixed domain and denote by  $c_P(\Omega)$  the smallest constant such that*

$$\|u\|_{\mathcal{L}^2(\Omega)}^2 \leq c_P(\Omega) \|\nabla_x u\|_{\mathcal{L}^2(\Omega)}^2$$

*hold for all  $u \in H_0^1(\Omega)$ . The constant  $c_P(\Omega)$  is called the Poincaré constant of the domain  $\Omega$ .*

**Proposition 2.2** (Interpolation-trace inequality). *Assume that  $Y \subset \mathbb{R}^d$  is a Lipschitz domain and  $u \in \mathcal{L}^2(\Omega; H^1(Y))$ . For any  $\rho > 0$  we have*

$$\int_{\Omega} \int_{\partial Y} u^2 d\sigma_y dx \leq \rho \int_{\Omega} \int_Y |\nabla_y u|^2 dy dx + c_{\rho} \int_{\Omega} \int_Y |u|^2 dy dx,$$

*where  $c_{\rho} \sim 1/\rho$ . In particular,*

$$\|u\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(\partial Y))} \leq c \|u\|_{\mathcal{L}^2(\Omega; H^1(Y))}.$$

The next result provides a useful equivalent norm on  $H^1$ .

**Proposition 2.3.** *Let  $U \subset \mathbb{R}^d$  be a domain and  $\Gamma \subset \partial Y$  where  $\Gamma$  has positive  $(d-1)$ -dimensional surface measure. Then there are constants  $c_1, c_2$  such that*

$$c_1 \|u\|_{H^1(U)}^2 \leq \int_{\Gamma} u^2 d\sigma + \|\nabla_x u\|_{\mathcal{L}^2(\Omega)}^2 \leq c_2 \|u\|_{H^1(U)}^2.$$

We shall also need the following two results, the first a Sobolev-type embedding and the second a simple trace theorem.

**Proposition 2.4.** *Assume that  $U \subset \mathbb{R}^d$ , then  $H^d(U) \subset \mathcal{L}^\infty(U)$  and*

$$\|u\|_{\mathcal{L}^\infty(U)} \leq C \|u\|_{H^d(U)}.$$

**Proposition 2.5.** *Assume that  $U \subset \mathbb{R}^d$  and  $\Gamma \subset \partial U$  is Lipschitz continuous. Then*

$$H^{d+1}(U) \subset H^d(\Gamma)$$

and

$$\|u\|_{H^d(\Gamma)} \leq c\|u\|_{H^{d+1}(U)}.$$

We have the following existence and regularity results.

**Proposition 2.6** (see e.g. Evans [1998]). *Consider the problem*

$$\begin{cases} \partial_t v - D\Delta_y v = 0 & \text{on } \Omega \times Y \\ -D\nabla_y v \cdot n_y = k(g - Rv) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y v \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ v(t=0) = v_I & \text{on } \Omega \times Y. \end{cases} \quad (2)$$

If  $g \in \mathcal{L}^2(0, T; H^1(\Omega))$  and  $v_I \in H^1(\Omega; H^m(Y))$  ( $m \in \mathbb{N}$ ), then the problem (2) has a unique weak solution  $v \in \mathcal{L}^2(0, T; H^1(\Omega; H^{m+1}(Y)))$ .

**Proposition 2.7** (see e.g. Cazenave [2006]). *Let  $v(t, x, y)$  be in  $\mathcal{L}^2(0, T; H^1(\Omega; H^{m+1}(Y)))$  ( $m \in \mathbb{N}$ ) and consider the problem*

$$\begin{cases} -\Delta_x u = f(u, v) & \text{on } \Omega \times Y \\ u(t, x) = 0 & \text{on } \partial\Omega, t > 0. \end{cases} \quad (3)$$

where the nonlinear function satisfies  $(A_4)$ – $(A_6)$ . Then the problem (3) has a unique weak solution  $u \in \mathcal{L}^2(0, T; H^1(\Omega))$ .

Finally, we state the following two classical compactness results, see e.g. Zeidler [1986].

**Theorem 2.8** (Aubin-Lions Theorem Aubin [1963]). *Let  $B_0 \hookrightarrow B \hookrightarrow B_1$ . Suppose that  $B_0$  is compactly embedded in  $B$  and that  $B$  is continuously embedded in  $B_1$ . Let*

$$W = \{u \in \mathcal{L}^2(0, T; B_0) : \partial_t u \in \mathcal{L}^2(0, T; B_1)\}. \quad (4)$$

*Then the embedding of  $W$  into  $\mathcal{L}^2(0, T; B)$  is compact.*

**Theorem 2.9** (Schauder's Fixed Point Theorem). *Let  $B$  be a nonempty, closed, convex, bounded set and  $T : B \rightarrow B$  a compact operator. Then there exists at least one  $r \in B$  such that  $T(r) = r$ .*

### 3. EXISTENCE AND UNIQUENES OF THE SOLUTION

**3.1. Existence of weak solution.** The main result of this subsection is the following theorem.

**Theorem 3.1.** *Assume that  $(A_2)$ – $(A_6)$  hold. Then the problem (1) has at least a weak solution  $(\pi, \rho) \in \mathcal{L}^2(0, T; H_0^1(\Omega)) \times \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; H^1(Y)))$ .*

*Proof.* We shall decouple the problem. The first sub-problem is as follows: given  $\pi \in \mathcal{L}^2(0, T; H_0^1(\Omega))$  and  $\rho_I \in H^1(\Omega, H^1(Y))$ , we let  $\xi$  be the weak solution to

$$\begin{cases} \partial_t \xi - D\Delta_y \xi = 0 & \text{on } \Omega \times Y \\ -D\nabla_y \xi \cdot n_y = k(\pi + p_F - R\xi) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y \xi \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ \xi(t=0) = \rho_I & \text{on } \Omega \times Y. \end{cases} \quad (5)$$

The weak formulation of (5) is: find  $\xi$  such that for a.e.  $t \in [0, T]$  and every  $\psi \in \mathcal{L}^2(\Omega, H^1(Y))$  there holds

$$\int_{\Omega} \int_Y \partial_t \xi \psi dy dx + \int_{\Omega} \int_Y D \nabla_y \xi \nabla_y \psi dy dx = \int_{\Omega} \int_{\Gamma_R} k(\pi + p_F - R\xi) \psi d\sigma_y dx, \quad (6)$$

and  $\xi(t = 0) = \rho_I$ . Existence and regularity of  $\xi$  is provided by Proposition 2.6 (recall that  $(A_3)$  states that  $\rho_I \in H^1(\Omega \times Y)$ ).

The second sub-problem is: given data  $\xi$ , consider the problem

$$\begin{cases} -\Delta_x \pi = f(\pi, \xi) & \text{on } \Omega \\ \pi = 0 & \text{on } \partial\Omega. \end{cases} \quad (7)$$

Let  $\lambda > 0$  be a free parameter. By the scaling properties of  $f$  and uniqueness of weak solution, we have that if  $\pi$  is the weak solution of (7) with data  $\xi$ , then  $\bar{\pi} = \lambda\pi$  is the weak solution to (7) with data  $\lambda\xi$ . Hence, if  $\bar{\pi}$  is the weak solution to

$$\begin{cases} -\Delta_x \bar{\pi} = \lambda^\beta f(\bar{\pi}, \xi) & \text{on } \Omega \\ \bar{\pi} = 0 & \text{on } \partial\Omega, \end{cases} \quad (8)$$

then  $\bar{\pi} = \lambda\pi$ , again by the scaling properties of  $f$ . The weak form of (8) is as follows: find  $\bar{\pi}$  such that for a.e.  $t \in [0, T]$  and all  $\varphi \in H_0^1(\Omega)$ , there holds

$$\int_{\Omega} \nabla_x \bar{\pi} \cdot \nabla_x \varphi dx = \lambda^\beta \int_{\Omega} f(\bar{\pi}, \xi) \varphi dx.$$

Existence and regularity of  $\bar{\pi}$  is guaranteed by Proposition 2.7.

We shall now use a fixed point argument à la Schauder (see Theorem 2.9) to show that there exists a  $\lambda > 0$  for which the functions of the pair  $(\bar{\pi}, \xi)$  are weak solutions to the sub-problems (5) and (8). Then we recover  $(\pi, \rho)$ , a weak solution to (1), by taking  $\pi = \bar{\pi}/\lambda$  and  $\rho = \xi$ .

Define the operators

$$T_1 : \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega)) \rightarrow \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y)))$$

by  $T_1(\pi) = \xi$  (the weak solution of (5)) and

$$T_2^\lambda : \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; H^1(Y))) \rightarrow \mathcal{L}^2(0, T; H^1(\Omega))$$

by  $T_2^\lambda(\xi) = \bar{\pi}$  (the weak solution of (8)). Finally, consider the operator  $\mathcal{A}^\lambda$  on the space  $\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega))$  into itself defined by

$$\mathcal{A}^\lambda(\pi) = T_2^\lambda(T_1(\pi)) = \bar{\pi}. \quad (9)$$

To obtain existence of solution, we shall prove that the operator  $\mathcal{A}^\lambda$  has a fixed point. This  $\pi$  will then give  $\xi$ . The idea of the proof is to first use the Schauder Fixed Point Theorem (Theorem 2.9 above).

We shall prove that there exist a  $\lambda > 0$  and a set  $B$  such that

- (1)  $\mathcal{A}^\lambda$  is a compact operator;
- (2)  $B$  is convex, closed, bounded and satisfies  $\mathcal{A}^\lambda(B) \subset B$ .

To obtain compactness of  $\mathcal{A}^\lambda = T_2^\lambda \circ T_1$ , it is sufficient to demonstrate that  $T_1$  is compact and that  $T_2^\lambda$  is continuous. Recall that we have

$$T_1 : \mathcal{L}^2(0, T; H^1(\Omega)) \rightarrow \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y))).$$

However, since we assume that  $\xi_I \in H^1(\Omega, H^1(Y))$  we get that  $T_1(\pi) = \xi \in \mathcal{L}^2(0, T; H^1(\Omega \times Y))$  and  $\partial_t \xi \in \mathcal{L}^2(0, T; L^2(\Omega \times Y))$ . Whence,

$$T_1(\mathcal{L}^2(0, T; H^1(\Omega))) \subset V,$$

where

$$V = \{u : u \in \mathcal{L}^2(0, T; H^1(\Omega \times Y)), \partial_t u \in \mathcal{L}^2(0, T; L^2(\Omega \times Y))\}$$

By Theorem 2.8,

$$V \hookrightarrow \hookrightarrow \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y))).$$

Thus, for any bounded set  $M \subset \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y))) \times \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega))$ , there holds  $T_1(M) \subset V$  and since  $V$  is compactly contained in  $\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y)))$  we have that  $T_1(M)$  is precompact in  $\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y)))$ . Hence,  $T_1$  is compact.

We continue to prove that  $T_2^\lambda$  is continuous. Assume we have two solutions  $\bar{\pi}_1 = T_2^\lambda(\xi_1)$  and  $\bar{\pi}_2 = T_2^\lambda(\xi_2)$ . Substituting these both in (9) and subtracting, we obtain

$$\int_{\Omega} \nabla_x(\bar{\pi}_1 - \bar{\pi}_2) \cdot \nabla_x \varphi dx = \lambda^\beta \int_{\Omega} [f(\bar{\pi}_1, \xi_1) - f(\bar{\pi}_2, \xi_2)] \varphi dx,$$

and for  $\varphi = \bar{\pi}_1 - \bar{\pi}_2$ , we get

$$\begin{aligned} \|\nabla_x(\bar{\pi}_1 - \bar{\pi}_2)\|_{\mathcal{L}^2(\Omega)}^2 &= \lambda^\beta \int_{\Omega} [f(\bar{\pi}_1, \xi_1) - f(\bar{\pi}_2, \xi_2)] [\bar{\pi}_1 - \bar{\pi}_2] dx \\ &= \lambda^\beta \int_{\Omega} [f(\bar{\pi}_1, \xi_1) - f(\bar{\pi}_2, \xi_1)] [\bar{\pi}_1 - \bar{\pi}_2] dx \\ &\quad + \lambda^\beta \int_{\Omega} [f(\bar{\pi}_2, \xi_1) - f(\bar{\pi}_2, \xi_2)] [\bar{\pi}_1 - \bar{\pi}_2] dx. \end{aligned}$$

Using  $(A_5)$  and  $(A_6)$ , we obtain that

$$\|\nabla_x(\bar{\pi}_1 - \bar{\pi}_2)\|_{\mathcal{L}^2(\Omega)}^2 \leq C \lambda^\beta \|\bar{\pi}_1 - \bar{\pi}_2\|_{\mathcal{L}^2(\Omega)} \|\xi_2 - \xi_1\|_{\mathcal{L}^2(\Omega; H^1(Y))}.$$

By the Poincaré's inequality, we obtain

$$\|\nabla_x(\bar{\pi}_1 - \bar{\pi}_2)\|_{\mathcal{L}^2(\Omega)} \leq C \lambda^\beta \|\xi_1 - \xi_2\|_{\mathcal{L}^2(\Omega; H^1(Y))}$$

and we conclude the mapping  $T_2^\lambda$  is continuous.

Let  $K > 0$  be a fixed number that we specify later and let  $B_K$  be the collection of functions  $u \in \mathcal{L}^2(0, T; H^1(\Omega))$  such that

$$\max\{\|u\|_{\mathcal{L}^2(0, T; L^2(\Omega))}, \|\nabla_x u\|_{\mathcal{L}^2(0, T; L^2(\Omega))}\} \leq K.$$

For each  $K > 0$ , the set

$$B_K \subset \mathcal{L}^2(0, T; H^1(\Omega))$$

is a convex, closed and bounded. We show that we may select  $K > 0$  and  $\lambda > 0$  such that

$$\mathcal{A}^\lambda(B_K) \subset B_K. \tag{10}$$

Note that  $T_1(B_K)$  is a bounded subset of  $\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y)))$ , with a bound depending only on  $K$ . In other words,

$$\|\xi\|_{\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y)))} \leq CK. \tag{11}$$

Indeed, this follows from the fact that  $T_1$  is a compact operator.

We proceed by observing that we may choose  $\lambda > 0$  such that if  $u \in B_K$  is arbitrary and  $v = T_2^\lambda(T_1(u))$ , then

$$\max\{\|v\|_{\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega))}, \|v\|_{\mathcal{L}^2(0, T; H^1(\Omega))}\} \leq K. \tag{12}$$

Let  $\xi = T_1(u)$  so that  $v = T_2^\lambda(\xi)$ . Testing the weak formulation of (8) with  $\varphi = u$  and using Cauchy-Schwarz' inequality and Poincaré's inequality, we get

$$\|\nabla_x u\|_{\mathcal{L}^2(\Omega)}^2 \leq C\lambda^\beta \|u\|_{\mathcal{L}^2(\Omega)} \|\xi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))} \leq CK\lambda^\beta \|\xi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}.$$

Integrating over  $[0, T]$  and using (11), we obtain after using Poincaré's inequality

$$\|u\|_{\mathcal{L}^2(0, T; H^1(\Omega))}^2 \leq C' K^2 \lambda^\beta.$$

By taking  $\lambda$  small enough (depending on  $K$ ), we obtain (12) whence (10) follows.  $\square$

*Remark 2.* Instead of using scaling arguments and Schauder's fixed point theorem, we could have used alternatively the Schaefer/Leray-Schauder fixed point theorem.

**3.2. Uniqueness of weak solutions.** We proceed to prove the following uniqueness theorem.

**Theorem 3.2.** *Assume that in addition to the assumptions of Theorem 3.1 the condition  $(A_7)$  also holds. Then the weak solution to (1) is unique.*

*Proof.* The weak formulation of the uncoupled problem is: find  $(\pi, \rho) \in \mathcal{L}^2(0, T; H_0^1(\Omega)) \times \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; H^1(Y)))$  where  $\rho(t=0) = \rho_I$  and for a.e.  $t \in [0, T]$  the equations

$$A\rho_F \int_{\Omega} \nabla_x \pi \cdot \nabla_x \varphi dx = \int_{\Omega} f(\pi, \rho) \varphi dx \quad (13)$$

and

$$\int_{\Omega} \int_Y \partial_t \rho \psi dy dx + \int_{\Omega} \int_Y D \nabla_y \rho \cdot \nabla_y \psi dy dx = \int_{\Omega} \int_{\Gamma_R} k(\pi + p_F - R\rho) \psi d\sigma_y dx \quad (14)$$

hold for all  $\varphi \in H_0^1(\Omega)$  and all  $\psi \in \mathcal{L}^2(\Omega; H^1(Y))$ .

Assume that two pairs of solutions exist:  $(\pi_1, \rho_1)$  and  $(\pi_2, \rho_2)$ . Let  $q := \pi_1 - \pi_2$  and  $z := \rho_1 - \rho_2$ . If we substitute the two solutions in (13) and (14) and subtract, we obtain that

$$A\rho_F \int_{\Omega} \nabla_x q \cdot \nabla_x \varphi dx = \int_{\Omega} (f(\pi_1, \rho_1) - f(\pi_2, \rho_2)) \varphi dx \quad (15)$$

and

$$\int_{\Omega} \int_Y \partial_t z \psi dy dx + \int_{\Omega} \int_Y D \nabla_y z \cdot \nabla_y \psi dy dx = \int_{\Omega} \int_{\Gamma_R} k(q - Rz) \psi d\sigma_y dx \quad (16)$$

for all  $\varphi \in H_0^1(\Omega)$  and all  $\psi \in \mathcal{L}^2(\Omega; H^1(Y))$ .

Choosing specific test function  $\varphi = q$ , using Young's inequality with parameter  $\varepsilon_1 > 0$  and  $(A_5)$ , we obtain from (15) the first key estimate

$$A\rho_F \|\nabla_x q\|_{\mathcal{L}^2(\Omega)}^2 \leq (C^* + \varepsilon_1) \|q\|_{\mathcal{L}^2(\Omega)}^2 + c_{\varepsilon_1} \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2. \quad (17)$$

We focus on (16), which, using test function  $\psi = z$ , yields

$$\frac{1}{2} \frac{d}{dt} \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + D \|\nabla_y z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 = \int_{\Omega} \int_{\Gamma_R} k(q - Rz) z. \quad (18)$$

Now, we estimate the right hand side of (18) by using trace inequality and the fact that  $k \leq \bar{k}$  on  $\Gamma_R$ . We have

$$\begin{aligned} \int_{\Omega} \int_{\Gamma_R} |k(q - Rz) z| d\sigma_y dx &\leq \bar{k} \int_{\Omega} \int_{\Gamma_R} |q z| d\sigma_y dx + R\bar{k} \int_{\Omega} \int_{\Gamma_R} z^2 d\sigma_y dx \\ &\leq \frac{\bar{k} |\Gamma_R|}{2} \|q\|_{\mathcal{L}^2(\Omega)}^2 + \left(R + \frac{1}{2}\right) \bar{k} \int_{\Omega} \int_{\Gamma_R} z^2 d\sigma_y dx. \end{aligned}$$

The second term at the right-hand side of the previous inequality can be estimated by using the trace inequality and Young's inequality with parameter  $\varepsilon > 0$ :

$$\int_{\Omega} \int_{\Gamma_R} z^2 d\sigma_y dx \leq \varepsilon \|\nabla_y z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + \frac{c_0}{\varepsilon} \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2$$

for some absolute constant  $c_0 > 0$ . Using the previous estimates and rearranging (18), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + (D - \varepsilon) \|\nabla_y z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 \\ & \leq \frac{\bar{k} |\Gamma_R|}{2} \|q\|_{\mathcal{L}^2(\Omega)}^2 + \frac{c_0 \bar{k} (R + 1/2)}{\varepsilon} \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2. \end{aligned} \quad (19)$$

By Poincaré's inequality, we have

$$\|q\|_{\mathcal{L}^2(\Omega)}^2 \leq c_P(\Omega) \|\nabla_x q\|_{\mathcal{L}^2(\Omega)}^2,$$

where  $c_P(\Omega)$  is the Poincaré constant of the domain  $\Omega$ . Using this in (17), we obtain

$$A\rho_F \|\nabla_x q\|_{\mathcal{L}^2(\Omega)}^2 \leq (C^* + \varepsilon_1) c_P(\Omega) \|\nabla_x q\|_{\mathcal{L}^2(\Omega)}^2 + C \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2.$$

By  $(A_7)$ , we may take  $\varepsilon_1 > 0$  small enough such that  $(C^* + \varepsilon_1) c_P(\Omega) < A\rho_F$ . Then we obtain

$$\|q\|_{\mathcal{L}^2(\Omega)}^2 \leq c_P(\Omega) \|\nabla_x q\|_{\mathcal{L}^2(\Omega)}^2 \leq \frac{C c_P(\Omega)}{A\rho_F - (C^* + \varepsilon_1) c_P(\Omega)} \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2. \quad (20)$$

Whence, it follows from the previous estimate and (19) with  $\varepsilon = D/2$  that

$$\frac{1}{2} \frac{d}{dt} \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + \frac{D}{2} \|\nabla_y z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 \leq C \|z\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2$$

By using Grönwall's inequality and the fact that  $z(0, x, y) = 0$ , it follows that  $z = 0$ . From (20), we obtain  $q = 0$  as well. This demonstrates the uniqueness.  $\square$

#### 4. ENERGY AND STABILITY ESTIMATES

We start this section by stating the following energy estimates for our problem.

**Proposition 4.1.** *Assume  $(A_2)$ – $(A_6)$  and let  $(u, v)$  be a weak solution to*

$$\begin{cases} -\Delta_x u = f(u, v) & \text{in } \Omega \\ \partial_t v - D\Delta_y v = 0 & \text{in } \Omega \times Y \\ -D\nabla_y v \cdot n_y + k(Rv - u) = g & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y v \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ u = 0 & \text{at } \partial\Omega \\ v(t=0) = v_I & \text{in } \Omega \times Y. \end{cases} \quad (21)$$

*Then the following energy estimate hold*

$$\begin{aligned} & \|u\|_{\mathcal{L}^2(0, T; H^1(\Omega))}^2 + \|v\|_{\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega, H^1(Y)))}^2 \\ & \leq C \left( \|g\|_{\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(\Gamma_R)))}^2 + \|v_I\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 \right). \end{aligned} \quad (22)$$

The proof of Proposition 4.1 follows by similar arguments as the proof of Theorem 4.2 below, therefore we omit it.

We proceed to study the stability of solutions with respect to some of the parameters involved. Some preliminary remarks:



- We do not need to study the stability of the solution with respect to  $\rho_F, p_F$  and  $R$ . Recall that  $R$  is an universal physical constant, while  $\rho_F, p_F$  fix the type of fluid and gas we are considering.
- We could investigate the stability of  $(\pi, \rho)$  with respect to structural changes into the non-linearity  $f(\cdot, \cdot)$ . We omit to do so mainly because our main intent lies in understanding the role of the micro-macro Robin coefficient  $k$ .
- For this stability proof, we decide to use a direct method which relies essentially on energy estimates; see e.g. Muntean [2009].

For  $i \in \{1, 2\}$ , let  $(\pi_i, \rho_i)$  be two weak solutions corresponding to the sets of data  $(\rho_{Ii}, A_i, D_i, k_i)$ , where  $\rho_{Ii}, A_i, D_i, k_i$  denote the initial data, diffusion coefficients and mass-transfer coefficients of the solution  $(\pi_i, \rho_i)$ . Denote

$$\delta u := u_2 - u_1 \quad \text{where} \quad u \in \{\pi, \rho, \rho_I, A, D, k\}.$$

**Theorem 4.2.** *Assume that for  $i = 1, 2$ ,  $(A_i, D_i)$  belongs to a fixed compact subset of  $\mathbb{R}^2$ , that  $k_i \in \mathcal{K}$  and that  $\|\rho_{Ii}\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))} \leq C$ . Let  $(\pi_i, \rho_i)$  ( $i = 1, 2$ ) be weak solutions to (1) corresponding to the choices of data above. Then the estimate*

$$\begin{aligned} & \|\delta\pi\|_{\mathcal{L}^2(0,T;H_0^1(\Omega))}^2 + \|\delta\rho\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^1(Y)))}^2 \\ & \leq c \left( \|\delta k\|_{\mathcal{L}^2(\Gamma_R)}^2 + |\delta A| + |\delta D| + \|\delta\rho_I\|_{\mathcal{L}^2(\Omega;\mathcal{L}^2(Y))}^2 \right), \end{aligned} \quad (23)$$

holds

*Proof.* We have for  $i = 1, 2$

$$A_i \rho_F \int_{\Omega} \nabla_x \pi_i \cdot \nabla_x \varphi dx = \int_{\Omega} f(\pi_i, \rho_i) \varphi dx$$

and

$$\int_{\Omega} \int_Y \partial_t \rho_i \psi dy dx + \int_{\Omega} \int_Y D_i \nabla_y \rho_i \cdot \nabla_y \psi dy dx = \int_{\Omega} \int_{\Gamma_R} k_i (\pi_i + p_F - R \rho_i) \psi d\sigma_y dx,$$

for all  $\varphi \in H_0^1(\Omega)$  and  $\psi \in \mathcal{L}^2(\Omega; H^1(Y))$ .

Subtracting the corresponding equations and then testing with  $\varphi := \pi_2 - \pi_1$  and  $\psi := \rho_2 - \rho_1$  gives:

$$\rho_F \left( A_2 \int_{\Omega} \nabla_x \pi_2 \cdot \nabla_x \varphi dx - A_1 \int_{\Omega} \nabla_x \pi_1 \cdot \nabla_x \varphi dx \right) = \int_{\Omega} (f(\pi_2, \rho_2) - f(\pi_1, \rho_1)) \varphi dx, \quad (24)$$

and

$$\begin{aligned} & \frac{d}{2dt} \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + \int_{\Omega} \int_Y (D_2 \nabla_y \rho_2 - D_1 \nabla_y \rho_1) \cdot \nabla_y \psi dy dx \\ & = \int_{\Omega} \int_{\Gamma_R} (k_2 (\pi_2 + p_F - R \rho_2) - k_1 (\pi_1 + p_F - R \rho_1)) \psi d\sigma_y dx. \end{aligned} \quad (25)$$

Regarding (24), note that

$$\begin{aligned} & A_2 \int_{\Omega} \nabla_x \pi_2 \cdot \nabla_x \varphi dx - A_1 \int_{\Omega} \nabla_x \pi_1 \cdot \nabla_x \varphi dx \\ & = A_2 \|\nabla_x \varphi\|_{\mathcal{L}^2(\Omega)}^2 + (A_2 - A_1) \int_{\Omega} \nabla_x \pi_1 \cdot \nabla_x \varphi dx \end{aligned}$$

Using (A<sub>5</sub>) and (A<sub>6</sub>), we may estimate the right-hand side of (24) and obtain

$$A_2 \rho_F \|\nabla_x \varphi\|_{\mathcal{L}^2(\Omega)}^2 \leq C^* \|\varphi\|_{\mathcal{L}^2(\Omega)}^2 + c \left( \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + |\delta A| \int_{\Omega} |\nabla_x \pi_1| |\nabla_x \varphi| dx \right).$$

Using Poincaré's inequality, assumptions on  $f$  and Young's inequality with parameter  $\varepsilon > 0$ , we get

$$\begin{aligned} \|\nabla_x \varphi\|_{\mathcal{L}^2(\Omega)}^2 &\leq c \left( \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + |\delta A| \int_{\Omega} |\nabla_x \pi_1| |\nabla_x \varphi| dx \right) \\ &\leq c\varepsilon \|\nabla_x \varphi\|_{\mathcal{L}^2(\Omega)}^2 + c \left( \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + |\delta A| \|\nabla_x \pi_1\|_{\mathcal{L}^2(\Omega)}^2 \right). \end{aligned}$$

Choosing  $\varepsilon = 1/(2c)$ , rearranging and using energy estimates for  $\pi_1$ , we obtain

$$\|\nabla_x \varphi\|_{\mathcal{L}^2(\Omega)}^2 \leq c \left( \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + |\delta A| \right). \quad (26)$$

We proceed to estimate  $\|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2$ , using (25). Note that

$$\begin{aligned} &\int_{\Omega} \int_Y (D_2 \nabla_y \rho_2 - D_1 \nabla_y \rho_1) \cdot \nabla_y \psi dy dx \\ &= (D_2 - D_1) \int_{\Omega} \int_Y \nabla_y \rho_2 \cdot \nabla_y \psi + D_1 \|\nabla_y \psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} &\frac{d}{dt} \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + D_1 \|\nabla_y \psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 \leq |\delta D| \int_{\Omega} \int_Y |\nabla_y \xi_1| |\nabla_y \psi| dy dx \\ &+ \int_{\Omega} \int_{\Gamma_R} |(k_2 - k_1)\psi| + |(k_2 \pi_2 - k_1 \pi_1)\psi| + |R(k_2 \rho_2 - k_1 \rho_1)\psi| d\sigma_y dx. \end{aligned} \quad (27)$$

We have

$$\begin{aligned} \int_{\Omega} \int_{\Gamma_R} |k_2 - k_1| |\psi| d\sigma_y dx &\leq \varepsilon \int_{\Omega} \int_{\Gamma_R} \psi^2 d\sigma_y dx + c_{\varepsilon} |\Omega| \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 \\ &\leq c\varepsilon \left( \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + \|\nabla_y \psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 \right) + c \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2. \end{aligned}$$

Further,

$$\begin{aligned} &\int_{\Omega} \int_{\Gamma_R} |(k_2 \pi_2 - k_1 \pi_1)\psi| d\sigma_y dx \leq c \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 + \int_{\Omega} \int_{\Gamma_R} |k_1| |\pi_2 - \pi_1| |\psi| d\sigma_y dx \\ &\leq c \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 + \varepsilon \bar{k} |\Gamma_R| \|\pi_2 - \pi_1\|_{\mathcal{L}^2(\Omega)}^2 + c_{\varepsilon} \int_{\Omega} \int_{\Gamma_R} \psi^2 d\sigma_y dx \\ &\leq c \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 + \varepsilon \left( \|\nabla_x \varphi\|_{\mathcal{L}^2(\Omega)}^2 + \|\nabla_y \psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 \right) + c_{\varepsilon} \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2. \end{aligned}$$

Finally,

$$R \int_{\Omega} \int_{\Gamma_R} |(k_2 \rho_2 - k_1 \rho_1)\psi| d\sigma_y dx \leq R \int_{\Omega} \int_{\Gamma_R} k_2^2 \psi^2 + (k_2 - k_1)^2 \rho_1^2 d\sigma_y dx.$$

We assume that for all  $y \in \Gamma_R$ , we have

$$\int_{\Omega} \rho_1^2 dx \leq K,$$

this can be ensured by taking  $\rho_{I1}$  smooth enough. Hence,

$$\begin{aligned} \int_{\Omega} \int_{\Gamma_R} k_2^2 \psi^2 + (k_2 - k_1)^2 \rho_1^2 d\sigma_y dx &\leq \bar{k} \int_{\Omega} \int_{\Gamma_R} \psi^2 d\sigma_y dx + K \|k_2 - k_1\|_{\mathcal{L}^2(\Omega)}^2 \\ &\leq \varepsilon \|\nabla_y \psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + c_\varepsilon \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + K \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2. \end{aligned}$$

Taking all the estimates above into consideration, and compensating terms by selecting small  $\varepsilon > 0$ , we finally obtain

$$\begin{aligned} \frac{d}{2dt} \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + D_1 \|\nabla_y \psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 \\ \leq c \left( \|\psi\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + |\delta D| + \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 \right) \end{aligned}$$

Applying Grönwall's inequality leads to

$$\|\psi(t)\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 \leq C \left[ \|\delta \rho_I\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + |\delta D| + \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 \right]$$

and, by integration over  $[0, T]$ ,

$$\|\psi\|_{\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y)))}^2 \leq CT \left[ \|\delta \rho_I\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + |\delta D| + \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 \right].$$

It also follows that

$$\|\nabla_y \psi\|_{\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y)))}^2 \leq CT^2 \left[ \|\delta \rho_I\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + |\delta D| + \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 \right].$$

Further, by (26) and Poincaré's inequality, we have

$$\|\varphi\|_{\mathcal{L}^2(0, T; H_0^1(\Omega))}^2 \leq C \left( \|\psi\|_{\mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(Y)))}^2 + |\delta A| \right).$$

Taking all the above estimates together, we obtain

$$\begin{aligned} \|\varphi\|_{\mathcal{L}^2(0, T; H_0^1(\Omega))}^2 + \|\psi\|_{\mathcal{L}^2(0, T; L^2(\Omega; H^1(Y)))}^2 \\ \leq C \left( |\delta A| + |\delta D| + \|\delta \rho_I\|_{\mathcal{L}^2(\Omega; \mathcal{L}^2(Y))}^2 + \|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)}^2 \right), \end{aligned}$$

which concludes the proof.  $\square$

## 5. LOCAL STABILITY FOR THE INVERSE ROBIN PROBLEM

In this section, we shall study the inverse problem of recovering the micro-macro Robin coefficient  $k \in \mathcal{L}^2(\Gamma_R)$  from measurement on  $\Gamma_N$ ; the Neumann part of the boundary. (Usually, one thinks of  $\Gamma_R$  as the inaccessible part of  $\partial Y$ , while  $\Gamma_N$  is the accessible part.) Our discussion is influenced by the work Jiang and Zou [2016]. An alternative way of working could be by following the abstract result in Bourgeois [2013].

Recall that we denote

$$\mathcal{K} = \{k \in \mathcal{L}^2(\Gamma_R) : 0 < \underline{k} \leq k(y) \leq \bar{k} \text{ for } y \in \Gamma_R\},$$

the set of admissible Robin coefficients. Denote by  $k^*$  the true Robin coefficient of our problem and define the set  $\mathcal{V}(k^*, a)$  as

$$\mathcal{V}(k^*, a) = \{k \in \mathcal{K} : \|k - k^*\|_{\mathcal{L}^2(\Gamma_R)} \leq a\}.$$

Below  $(\pi(k), \rho(k))$  denotes the solution to (1) corresponding to the coefficient  $k \in \mathcal{K}$ . Our main result is the following theorem.

**Theorem 5.1.** *Assume that  $\rho_I \in H^1(\Omega, H^d(Y))$  and  $\rho(k^*) \geq c_0 > 0$  on  $[0, T] \times \Omega \times \Gamma_R$ . Then there exists a  $a > 0$  such that*

$$\|\rho(k_2) - \rho(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_N)))} \geq c\|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)} \quad (28)$$

for every  $k_1, k_2 \in \mathcal{V}(k^*, a)$ .

*Remark 3.* The discussion around Theorem 5.1 can be extended to the case of recovering micro-macro Robin coefficient with a genuine two-scale structure, e.g.  $k \in \mathcal{L}^2(\Omega; \mathcal{L}^2(\Gamma_R))$  or  $k \in \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(\Gamma_R)))$ . In this case, two-scale measurements are needed. To keep the presentation as simple as possible, we focus our attention on  $k \in \mathcal{K}$ .

In the rest of this section, we prove establish several lemmata. The proof of Theorem 5.1 is given in Section 6.

**Lemma 5.2.** *For any  $k \in \mathcal{K}$  and  $d \in \mathcal{L}^\infty(\Gamma_R)$  let  $(\pi(k), \rho(k))$  be the solution to (1) and  $(u, v) = (u(k), v(k))$  the solution to*

$$\begin{cases} -\Delta_x u = F(u, v) & \text{in } \Omega \\ \partial_t v - D\Delta_y v = 0 & \text{in } \Omega \times Y \\ -D\nabla_y v \cdot n_y + k(Rv - u) = d(\pi(k) - p_F - R\rho(k)) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y v \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ u = 0 & \text{at } \partial\Omega \\ v(0, x, y) = 0 & \text{in } \Omega \times Y, \end{cases} \quad (29)$$

where  $F(u, v)$  is specified below. Then  $\rho(k)$  is continuously Fréchet differentiable and its derivative  $\rho'(k)d$  at  $d \in \mathcal{L}^\infty(\Gamma_R)$  is given by  $v(k)$ .

*Proof.* One can observe that the well-posedness of (29) follows by similar arguments as in the previous sections. Take  $k \in \mathcal{K}$  and  $d \in \mathcal{L}^\infty(\Gamma_R)$  such that  $k + d \in \mathcal{K}$ .

Note first that  $(u_1(k), v_1(k)) = (\pi(k + d) - \pi(k), \rho(k + d) - \rho(k))$  solves

$$\begin{cases} -\Delta_x u_1 = f_1(u_1, v_1) & \text{in } \Omega \\ \partial_t v_1 - D\Delta_y v_1 = 0 & \text{in } \Omega \times Y \\ -D\nabla_y v_1 \cdot n_y + k(Rv_1 - u_1) = d(\pi(k + d) - p_F - R\rho(k + d)) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y v_1 \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ u_1 = 0 & \text{at } \partial\Omega \\ v_1(0, x, y) = 0 & \text{in } \Omega \times Y, \end{cases}$$

where  $f_1(u_1, v_1) = f(\pi(k + d), \rho(k + d)) - f(\pi(k), \rho(k))$ . Denote by  $F = f_1$  and

$$U = u_1 - u = \pi(k + d) - \pi(k) - u, \quad V = v_1 - v = \rho(k + d) - \rho(k) - v,$$

then  $(U, V)$  solves the problem

$$\begin{cases} -\Delta_x U = f_2(U, V) & \text{in } \Omega \\ \partial_t V - D\Delta_y V = 0 & \text{in } \Omega \times Y \\ -D\nabla_y V \cdot n_y + k(RV - U) = d(u_1 - Rv_1) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y V \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ U = 0 & \text{at } \partial\Omega \\ V(0, x, y) = 0 & \text{in } \Omega \times Y, \end{cases}$$

where  $f_2(U, V) = f_1(u_1, v_1) - f_1(u, v)$ . Note that the nonlinearities  $f_1, f_2$  satisfy the conditions of the energy estimate Proposition 4.1. Thus,

$$\|V\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,H^1(Y)))}^2 \leq C\|d\|_{\mathcal{L}^\infty(\Gamma_R)}^2\|u_1 - Rv_1\|_{\mathcal{L}^2(\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))}^2.$$

Whence,

$$\begin{aligned} \frac{\|\rho(k+d) - \rho(k) - v\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,H^1(Y)))}}{\|d\|_{\mathcal{L}^\infty(\Gamma_R)}} &\leq \\ &\leq C \left( \|u_1\|_{\mathcal{L}^2(\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} + \|v_1\|_{\mathcal{L}^2(\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} \right), \end{aligned}$$

and it is sufficient to show that the right-hand side above tends to 0 as  $\|d\|_{\mathcal{L}^\infty(\Gamma_R)} \rightarrow 0$ . Using the interpolation-trace inequality, we obtain

$$\|v_1\|_{\mathcal{L}(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} \leq C\|v_1\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,H^1(Y)))}.$$

Furthermore, using Proposition 4.1 and the interpolation-trace inequality again, we obtain

$$\begin{aligned} &\|u_1\|_{\mathcal{L}^2(0,T;H^1(\Omega))}^2 + \|v_1\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,H^1(Y)))}^2 \leq \\ &C\|d\|_{\mathcal{L}^\infty(\Gamma_R)}^2\|\pi(k+d) - p_F - R\rho(k+d)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))}^2 \leq C'\|d\|_{\mathcal{L}^\infty(\Gamma_R)}^2 \end{aligned}$$

from which follows that

$$\lim_{d \rightarrow 0} \frac{\|\rho(k+d) - \rho(k) - v\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,H^1(Y)))}}{\|d\|_{\mathcal{L}^\infty(\Gamma_R)}} = 0.$$

The proof of continuity follows by a similar argument, we refer to the discussion in Jiang and Zou [2016].  $\square$

We proceed now in a similar fashion as in e.g. Choulli [2004], Jiang and Zou [2016].

Let  $g \in \mathcal{L}^2(\Gamma_R)$  and  $(\theta, \omega) = (\theta(g), \omega(g))$  be the weak solution to the system

$$\begin{cases} -\Delta_x \theta = f(\theta, \omega) & \text{in } \Omega \\ \partial_t \omega - D\Delta_y \omega = 0 & \text{in } \Omega \times Y \\ -D\nabla_y \omega \cdot n_y + k^* R\omega = -g\rho(k^*) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y \omega \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ \theta = 0 & \text{at } \partial\Omega \\ \omega(0, x, y) = 0 & \text{in } \Omega \times Y \end{cases} \quad (30)$$

For  $g \in \mathcal{L}^2(\Gamma_R)$ , define the operator

$$N : \mathcal{L}^2(\Gamma_R) \rightarrow \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(\Gamma_R)))$$

by

$$N(g) = -D\nabla_y \omega(g) \cdot n_y.$$

Then  $N$  is a bounded linear operator (boundedness follow from energy estimates).

**Lemma 5.3.** *The operator  $N$  is bijective and  $\|N^{-1}\|$  is finite.*

*Proof.* We prove the surjectivity of  $N$ . Assume that  $\varphi \in \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(\Gamma_R)))$ , we must prove that there exists  $g \in \mathcal{L}^2(\Gamma_R)$  such that  $N(g) = \varphi$ . Using (30), we obtain

$$\varphi + k^* R\omega(g) = -g\rho(k^*),$$

or, equivalently,

$$\frac{\varphi}{\rho(k^*)} + g = -\frac{k^* R\omega(g)}{\rho(k^*)}. \quad (31)$$

Define

$$\mathcal{O} : \mathcal{L}^2(\Gamma_R) \rightarrow \mathcal{L}^2(0, T; \mathcal{L}^2(\Omega; \mathcal{L}^2(\Gamma_R))).$$

by

$$\mathcal{O}(g) = -\frac{k^* R \omega(g)}{\rho(k^*)}$$

Then we have

$$\frac{\varphi}{\rho(k^*)} = (\mathcal{O} - I)(g).$$

Note further that  $\mathcal{O} = BA$ , where

$$A : g \mapsto \omega(g), \quad B : q \mapsto -\frac{k^* R q}{\rho(k^*)}.$$

We have seen that  $A$  is compact and  $B$  is clearly continuous. Hence,  $\mathcal{O}$  is compact.

We claim now that 1 is not an eigenvalue to  $\mathcal{O}$ . Then, by the Fredholm alternative theorem,  $\mathcal{O} - I$  is invertible and

$$g = (\mathcal{O} - I)^{-1} \left( \frac{\varphi}{\rho(k^*)} \right).$$

To prove that 1 is not an eigenvalue of  $\mathcal{O}$ , assume that  $\mathcal{O}(g) = g$  for some  $g \in \mathcal{L}^2(\Gamma_R)$ . It follows from (31) that  $\varphi/\rho(k^*) = 0$ , so  $\varphi = N(g) = 0$ . Hence,  $-D\nabla_y \omega(g) \cdot n_y = 0$  on  $[0, T] \times \Omega \times \Gamma_R$ . Since  $\omega(g)$  solves (30) it also solves

$$\begin{cases} -\Delta_x \theta = f(\theta, \omega) & \text{in } \Omega \\ \partial_t \omega - D\Delta_y \omega = 0 & \text{in } \Omega \times Y \\ -D\nabla_y \omega \cdot n_y = 0 & \text{on } \Omega \times \partial Y \\ \theta = 0 & \text{at } \partial\Omega \\ \omega(0, x, y) = 0 & \text{in } \Omega \times Y. \end{cases} \quad (32)$$

Hence,  $\omega(g) = 0$ , but then  $-g\rho(k^*) = 0$  from the Robin boundary condition of (30), and since  $\rho(k^*) \geq c_0 > 0$ , we get  $g = 0$ . In other words, 1 is not an eigenvalue of  $\mathcal{O}$ . In conclusion,  $N$  is invertible. Since  $N$  is bounded, bijective and linear, the open mapping theorem ensures that  $N^{-1}$  exists and is bounded.  $\square$

## 6. PROOF OF THEOREM 5.1

We are now ready to prove our main result.

*Proof of Theorem 5.1.* Let  $\varepsilon > 0$  and consider the scaled problem

$$\begin{cases} -\Delta_x \xi = f(\xi, \zeta) & \text{in } \Omega \\ \partial_t \zeta - D\Delta_y \zeta = 0 & \text{in } \Omega \times Y \\ -D\nabla_y \zeta \cdot n_y + kR\zeta = k(\xi + \varepsilon p_F) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y \zeta \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ \xi = 0 & \text{at } \partial\Omega \\ \zeta(0, x, y) = \varepsilon \rho_I & \text{in } \Omega \times Y. \end{cases} \quad (33)$$

Recall that we have  $f(\varepsilon u, \varepsilon v) = \varepsilon f(u, v)$ . From this it follows that the solution  $(\xi^\varepsilon, \zeta^\varepsilon)$  to the above problem satisfies  $(\xi^\varepsilon, \zeta^\varepsilon) = (\varepsilon \pi, \varepsilon \rho)$ . Note that  $\zeta^\varepsilon(k) = \varepsilon \zeta(k) \geq \varepsilon c_0 > 0$  on  $[0, T] \times \Omega \times \Gamma_R$ .

Define the norm  $\|\cdot\|_\varepsilon$  on  $\mathcal{L}^2(\Gamma_R)$  by

$$\|g\|_\varepsilon = \left\| \frac{1}{\zeta^\varepsilon(k^*)} g \right\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,\mathcal{L}^2(\Gamma_R)))}.$$

Further, define the mapping

$$\sigma_\varepsilon : \mathcal{K} \rightarrow \mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R))),$$

by  $\sigma_\varepsilon(k) = D\nabla_y(\zeta^\varepsilon) \cdot n_y$ . It follows from the fact that  $\zeta^\varepsilon(k)$  is Frechet differentiable with continuous derivative that  $\sigma_\varepsilon$  is a  $C^1$ -diffeomorphism. We have

$$\sigma'_\varepsilon(k)g = D\nabla_y\zeta_1^\varepsilon(k^*,g) \cdot n_y \quad (34)$$

where  $\zeta_1^\varepsilon(k^*,g)$  is the solution to

$$\begin{cases} -\Delta_x \xi_1^\varepsilon = f(\xi_1^\varepsilon, \zeta_1^\varepsilon(k^*,g)) & \text{in } \Omega \\ \partial_t \zeta_1^\varepsilon(k^*,g) - D\Delta_y \zeta_1^\varepsilon(k^*,g) = 0 & \text{in } \Omega \times Y \\ -D\nabla_y \zeta_1^\varepsilon(k^*,g) \cdot n_y + kR\zeta_1^\varepsilon(k^*,g) = -g\zeta^\varepsilon(k^*) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y \zeta_1^\varepsilon(k^*,g) \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ \xi_1^\varepsilon = 0 & \text{at } \partial\Omega \\ \zeta_1^\varepsilon(k^*,g)(0,x,y) = 0 & \text{in } \Omega \times Y. \end{cases}$$

Since  $\zeta^\varepsilon(k^*) = \varepsilon\rho(k^*)$ , it follows from (30) that  $\zeta_1^\varepsilon(k^*,g) = \varepsilon\omega(g)$  and by  $\sigma'_\varepsilon(k^*)g = \varepsilon N$  by (34). It follows that  $(\sigma'_\varepsilon(k^*)g)^{-1} = N^{-1}/\varepsilon$ . Since  $\sigma'_\varepsilon(k^*)g$  is a  $C^1$ -diffeomorphism, there exists a neighbourhood  $N(k^*,a)$  such that for any  $k_1, k_2 \in N(k^*,a)$  we have

$$\|k_2 - k_1\|_\varepsilon \leq 2\|(\sigma_\varepsilon(k^*)g^{-1})'\| \|\sigma'_\varepsilon(k_2)g - \sigma'_\varepsilon(k_1)g\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Gamma_R))}$$

(see the discussion in Jiang and Zou [2016]). We have

$$\begin{aligned} \|k_2 - k_1\|_\varepsilon &\leq 2\|(\sigma_\varepsilon(k^*)g^{-1})'\| \|D\nabla_y\zeta^\varepsilon(k_2) \cdot n_y - D\nabla_y\zeta^\varepsilon(k_1) \cdot n_y\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} \\ &\leq \frac{C}{\varepsilon} \|D\nabla_y\zeta^\varepsilon(k_2) \cdot n_y - D\nabla_y\zeta^\varepsilon(k_1) \cdot n_y\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))}. \end{aligned}$$

Using (33), one obtains the estimate

$$\begin{aligned} &\|D\nabla_y\zeta^\varepsilon(k_2) \cdot n_y - D\nabla_y\zeta^\varepsilon(k_1) \cdot n_y\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} \\ &\leq C (\|\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} + \|\xi^\varepsilon(k_2) - \xi^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))}) \\ &\quad + C \sum_{j=1}^2 \|\zeta^\varepsilon(k_j)(k_2 - k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))}. \end{aligned}$$

By the interpolation-trace inequality, we have

$$\|\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} \leq C \|\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^1(Y)))}.$$

Further, by the Poincaré inequality,

$$\|\xi^\varepsilon(k_2) - \xi^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))} \leq C \|\xi^\varepsilon(k_2) - \xi^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;H^1(\Omega))}.$$

Hence, we obtain

$$\begin{aligned} \|k_2 - k_1\|_\varepsilon &\leq \frac{C}{\varepsilon} \left[ \|\xi^\varepsilon(k_2) - \xi^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;H^1(\Omega))} + \|\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^1(Y)))} \right. \\ &\quad \left. + \sum_{j=1}^2 \|\zeta^\varepsilon(k_j)(k_2 - k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} \right]. \end{aligned}$$

We have

$$\sum_{j=1}^2 \|\zeta^\varepsilon(k_j)(k_2 - k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} \leq C \|k_2 - k_1\|_\varepsilon \sum_{j=1}^2 \|\zeta^\varepsilon(k_j)\zeta^\varepsilon(k^*)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^\infty(\Gamma_R)))}$$

Since  $\Gamma_R$  is  $(d-1)$ -dimensional, we have

$$\|\zeta^\varepsilon(k)\|_{\mathcal{L}^\infty(\Gamma_R)} \leq c \|\zeta^\varepsilon(k)\|_{H^{d-1}(\Gamma_R)} \leq c \|\zeta^\varepsilon(k)\|_{H^d(Y)}$$

by Proposition 2.5 and Proposition 2.4. Further, by Proposition 2.6 and the assumption  $\rho_I \in \mathcal{L}^2(\Omega; H^{d-1}(Y))$ , we have  $\zeta^\varepsilon(k) \in \mathcal{L}^2(0,T; H^1(\Omega; H^d(Y)))$ . Hence,

$$\begin{aligned} \|\zeta^\varepsilon(k^*)\zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^\infty(\Gamma_R)))} &\leq c \|\zeta^\varepsilon(k^*)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^{d-1}(\Gamma_R)))} \|\zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^{d-1}(\Gamma_R)))} \\ &\leq c \|\zeta^\varepsilon(k^*)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^d(Y)))} \|\zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^d(Y)))} \\ &\leq c\varepsilon^2 \|\rho(k^*)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^d(Y)))} \|\rho(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^d(Y)))} \\ &\leq C'\varepsilon^2. \end{aligned}$$

By the above estimates, we can rely on

$$(1 - C\varepsilon) \|k_2 - k_1\|_\varepsilon \leq \frac{C}{\varepsilon} \left( \|\xi^\varepsilon(k_2) - \xi^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;H^1(\Omega))} + \|\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^1(Y)))} \right).$$

Using the equivalent norm on  $\mathcal{L}^2(0,T; \mathcal{L}^2(\Omega; H^1(Y)))$  given by Proposition 2.3, we obtain

$$\begin{aligned} (1 - C\varepsilon) \|k_2 - k_1\|_\varepsilon &\leq \frac{C}{\varepsilon} \left( \|\xi^\varepsilon(k_2) - \xi^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;H^1(\Omega))} + \|\nabla_y(\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1))\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^1(Y)))} \right) \\ &\quad + \frac{C}{\varepsilon} \|\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_N)))}. \end{aligned}$$

Set  $X^\varepsilon := \xi^\varepsilon(k_2) - \xi^\varepsilon(k_1)$  and  $Z^\varepsilon := \zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)$ , then  $(X^\varepsilon, Z^\varepsilon)$  solves

$$\begin{cases} -\Delta_x X^\varepsilon = f_1(X^\varepsilon, Z^\varepsilon) & \text{in } \Omega \\ \partial_t Z^\varepsilon - D\Delta_y Z^\varepsilon = 0 & \text{in } \Omega \times Y \\ -D\nabla_y Z^\varepsilon \cdot n_y + k_1 R Z^\varepsilon = (k_2 - k_1)\zeta^\varepsilon(k_2) & \text{on } \Omega \times \Gamma_R \\ -D\nabla_y Z^\varepsilon \cdot n_y = 0 & \text{on } \Omega \times \Gamma_N \\ X^\varepsilon = 0 & \text{at } \partial\Omega \\ Z^\varepsilon(0, x, y) = 0 & \text{in } \Omega \times Y, \end{cases}$$

where  $f_1(X^\varepsilon, Z^\varepsilon) = f(\xi^\varepsilon(k_2), \zeta^\varepsilon(k_2)) - f(\xi^\varepsilon(k_1), \zeta^\varepsilon(k_1))$ .

Using Proposition 4.1 and similar estimates as above, we obtain

$$\begin{aligned} &\|\xi^\varepsilon(k_2) - \xi^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;H^1(\Omega))} + \|\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;H^1(Y)))} \\ &\leq C \|(k_2 - k_1)\zeta^\varepsilon(k_2)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega;\mathcal{L}^2(\Gamma_R)))} \leq C\varepsilon^2 \|k_2 - k_1\|_\varepsilon. \end{aligned}$$



This finally yields the crucial estimate

$$(1 - C'\varepsilon)\|k_2 - k_1\|_\varepsilon \leq \frac{C}{\varepsilon}\|\zeta^\varepsilon(k_2) - \zeta^\varepsilon(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,\mathcal{L}^2(\Gamma_N)))}.$$

Choose  $\varepsilon^* > 0$  such that  $1 - C'\varepsilon^* = 1/2$  and use that  $\zeta^\varepsilon = \varepsilon\rho$ , then we obtain

$$\|k_2 - k_1\|_{\varepsilon^*} \leq C\|\rho(k_2) - \rho(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,\mathcal{L}^2(\Gamma_N)))}.$$

Finally, since  $\zeta^*(k) \geq \varepsilon c_0$ , we obtain that

$$\|k_2 - k_1\|_{\mathcal{L}^2(\Gamma_R)} \leq C\|k_2 - k_1\|_{\varepsilon^*} \leq C\|\rho(k_2) - \rho(k_1)\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega,\mathcal{L}^2(\Gamma_N)))}.$$

□

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KARLSTAD UNIVERSITY, 651 88 KARLSTAD, SWEDEN

*E-mail address:* `martin.lind@kau.se`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KARLSTAD UNIVERSITY, 651 88 KARLSTAD, SWEDEN

*E-mail address:* `adrian.muntean@kau.se`

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KARLSTAD UNIVERSITY, 651 88 KARLSTAD, SWEDEN

*E-mail address:* `omar.richardson@kau.se`